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Generalized coherent states for associated hypergeometric-type functions

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Abstract

We introduce a right inverse for the annihilation operator which appears in the factorization of associated hypergeometric-type operators. This makes possible the definition of generalized coherent states for the associated eigenfunctions. These generalized coherent states satisfy the relevant properties, i.e., label continuity, overcompleteness, temporal stability and action identity required to establish a close connection between the quantum and classical formulations of a given physical system.

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1. Introduction

Coherent states continue to be at the core of many investigations [1–5]. A good compilation of relevant references on this topic is available in recent works by Antoine *et al* [6, 7]. Indeed, coherent states were first introduced in 1926 by Schrödinger who considered special quantum states that were particularly adapted for studying the quantum-to-classical transition. The term *coherent* was introduced in the 1960s by Glauber [8], Klauder [9], Sudarshan [10] and others, in the context of quantum optical description of coherent light beams emitted by lasers. The coherent states (CS) directly related to the canonical commutation relations were applied originally to the harmonic oscillator system. It was then noted by Gilmore [11] and Perelomov [12], independently, that CS were in fact closely related to a representation of the underlying group, namely, the Weyl–Heisenberg group. Most of the interesting properties of those canonical CS derive from the square integrability of that representation. This leads to the extension of the concept of CS for a number of Lie groups with square integrable representations. Today CS are widely used in different fields of physics and mathematics [13–15], and in several applications [16, 17]. In fact, there exist various definitions of CS.

Recently, Aleixo and Balantekin [18] have pointed out three main definitions. The first one, often called Barut–Girardello CS [19], assumes that the CS are eigenstates with complex eigenvalues of an annihilation group operator. The second definition, often called Perelomov CS [12], assumes the existence of a unitary z-displacement operator, the action of which on the ground state of the system gives the CS parametrized by z, with $z \in \mathbb{C}$. The last definition, based on the Heisenberg uncertainty relation, often called *intelligent* CS [20, 21], assumes that the CS gives the minimum-uncertainty value $\Delta x \Delta p = \frac{\hbar}{2}$ and maintains this relation in time because of its temporal stability. Thus, the first definition makes connection with the supersymmetry quantum mechanics (SUSY QM).

SUSY QM is essentially the study of partner Hamiltonians which are isospectral, that is, they have almost the same energy eigenvalues [22–25]. A number of such partner Hamiltonians verify an integrability criterion known as shape invariance [26]. Although not all exactly solvable problems are shape invariant, shape invariance, especially in its algebraic formulation, seems to be the decisive test for the solvability of a given SUSY system. It was shown [27] that with the shape-invariance condition are associated specific Lie algebras and CS defined as eigenstates of the annihilation factorizing operator [28–31]. In this connection, Aleixo and Balantekin [18] have provided a definition of generalized CS for shape-invariant potentials and showed that these states fulfil the properties of label continuity, resolution of unity, temporal stability and action identity, required to establish a close connection between classical and quantum formulations of a given system.

The factorization method [32, 33], initially developed for the Schrödinger operator to split it into a creation and an annihilation operator, has been now extended to the more generalized second-order differential operators of mathematical physics. Indeed, in a recent work [34], we have proved that the common widely used factorization scheme can be extended to the more general Sturm–Liouville operators. Jafarizadeh and Fakhri [35] have factorized some types of differential operators and deduced the corresponding shape-invariance relations. Cotfas [36], following Jafarizadeh and Lorente [37], has provided a way of factorization of associated hypergeometric-type operators and deduced the corresponding CS. Aleixo and Balantekin [18] have constructed the generalized CS for shape-invariant potentials using an algebraic approach based on SUSY QM. Our paper aims at generalizing the work of these authors to the associated hypergeometric-type operators.

In section 2, we give a brief review of the factorization of associated hypergeometric-type operators and the construction of the corresponding CS following Cotfas [36]. In section 3, we provide a generalization of the CS. These states satisfy the standard properties of label continuity, overcompleteness, temporal stability and action identity. In section 4, some examples are given. Finally, we end with a concluding section.

2. Brief review of the factorization of associated hypergeometric-type operators

Classical orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$ satisfy the differential equation [38]

$$\sigma(s)\Phi_l''(s) + \tau(s)\Phi_l'(s) + \lambda_l\Phi_l(s) = 0 \tag{1}$$

where $\lambda_l = -\frac{1}{2}l(l+1)\sigma'' - l\tau'$, σ and τ are polynomials of at most second and exactly first degrees, respectively. They are orthogonal with respect to the non-negative weight function ρ,

$$\int_a^b \Phi_l(s)\Phi_k(s)\rho(s)\,\mathrm{d}(s) = 0 \qquad \text{for} \quad l \neq k, \tag{2}$$

where ρ satisfies Pearson's equation $(\sigma\rho)' = \tau\rho$, over the interval (a, b), which can be finite or infinite, and further satisfy

$$\sigma(s)\rho(s)s^k|_{s=a} = \sigma(s)\rho(s)s^k|_{s=b} = 0 \qquad \text{for all} \quad k \in \mathbb{N}.$$

Differentiating (1) m times and multiplying by κ^m, with $\kappa = \sqrt{\sigma}$, we get the eigenvalue problem $H_m\Phi_{l,m} = \lambda_l\Phi_{l,m}$, where

$$H_m = -\sigma\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \tau\frac{\mathrm{d}}{\mathrm{d}s} + \frac{m(m-2)}{4}\frac{\sigma'^2}{\sigma} + \frac{m}{2}\tau\frac{\sigma'}{\sigma} - \frac{1}{2}m(m-2)\sigma'' - m\tau' \tag{3}$$

and $\Phi_{l,m} = \kappa^m\Phi_l^{(m)}$ are what we call associated hypergeometric-type functions (AHF). One can show that these functions are orthogonal with respect to ρ, that is

$$\int_a^b \Phi_{l,m}\Phi_{k,m}\rho\,\mathrm{d}s = 0, \qquad l \neq k, \qquad l, k \in \{m, m+1, m+2, \ldots\}. \tag{4}$$

Let $\mathcal{H}_m$ be the Hilbert space of $\{\Phi_{k,m}\}_{k\geqslant m}$ (for a given $m \in \mathbb{N}$) with respect to the inner product (4). This space coincides with the Hilbert space

$$\mathcal{H} = \left\{\varphi : (a, b) \longrightarrow \mathbb{C} \Big/ \int_a^b |\varphi(s)|^2\rho(s)\,\mathrm{d}s < \infty\right\}.$$

Let

$$A_m : \mathcal{H}_m \longrightarrow \mathcal{H}_{m+1} \qquad \text{and} \qquad A_m^\dagger : \mathcal{H}_{m+1} \longrightarrow \mathcal{H}_m$$

be mutually adjoint first-order differential operators defined as in [36]

$$A_m = \kappa(s)\frac{\mathrm{d}}{\mathrm{d}s} - m\kappa'(s) \qquad \text{and} \qquad A_m^\dagger = -\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s} - \frac{\tau(s)}{\kappa(s)} - (m-1)\kappa'(s).$$

The operator $H_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m$ factorizes as

$$H_m - \lambda_m = A_m^\dagger A_m, \qquad H_{m+1} - \lambda_m = A_m A_m^\dagger.$$

The operator H_m then fulfils the intertwining relations

$$H_m A_m^\dagger = A_m^\dagger H_{m+1} \qquad \text{and} \qquad A_m H_m = H_{m+1}A_m.$$

One can deduce the following shape-invariance relations:

$$A_m A_m^\dagger = A_{m+1}^\dagger A_{m+1} + r_{m+1}, \qquad r_{m+1} = \lambda_{m+1} - \lambda_m = -m\sigma'' - \tau', \tag{5}$$

and eigenvalues λ_l and eigenfunctions $\Phi_{l,m}$ as

$$\lambda_l = \sum_{k=1}^{l} r_k, \qquad \Phi_{l,m} = \frac{A_m^\dagger}{\lambda_l - \lambda_m}\frac{A_{m+1}^\dagger}{\lambda_l - \lambda_{m+1}}\cdots\frac{A_{l-2}^\dagger}{\lambda_l - \lambda_{l-2}}\frac{A_{l-1}^\dagger}{\lambda_l - \lambda_{l-1}}\Phi_{l,l} \tag{6}$$

for all $l \in \mathbb{N}$ and $m \in \{0, 1, 2, \ldots, l-1\}$ where $\Phi_{l,l}$ satisfies the relation $A_l\Phi_{l,l} = 0$. Introducing, for each $m \in \mathbb{N}$, the sequence $\{|m, m\rangle, |m+1, m\rangle, \ldots\}$, where $|l, m\rangle = \dfrac{\phi_{l,m}}{\|\Phi_{l,m}\|}$, we can define a unitary operator $U_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m, U_m|l, m\rangle = |l+1, m+1\rangle$ and an annihilation and a creation operator

$$a_m, a_m^\dagger : \mathcal{H}_m \longrightarrow \mathcal{H}_m, \qquad a_m = U_m^\dagger A_m \qquad \text{and} \qquad a_m^\dagger = A_m^\dagger U_m.$$

The operators a_m and $a_m^\dagger$ are mutually adjoint and act on the state $|l, m\rangle$ as

$$a_m|l, m\rangle = \sqrt{\lambda_l - \lambda_m}|l-1, m\rangle \qquad \text{and} \qquad a_m^\dagger|l, m\rangle = \sqrt{\lambda_{l+1} - \lambda_m}|l+1, m\rangle, \qquad l \geqslant m,$$

with the algebra

$$\left[a_m, a_m^\dagger\right] = \mathcal{R}_m, \qquad \left[a_m^\dagger, \mathcal{R}_m\right] = \sigma'' a_m^\dagger \qquad \text{and} \qquad [a_m, \mathcal{R}_m] = -\sigma'' a_m \tag{7}$$

where $\mathcal{R}_m = -\sigma'' N_m - \tau'$, $N_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m$ is the number operator defined as $N_m \Phi_{l,m} = l\Phi_{l,m}$. In addition to the commutation relations (7) we have

$$A_m \mathcal{R}_m = \mathcal{R}_{m+1} A_m \tag{8}$$

and the similarity transformation relation

$$U_m \mathcal{R}_m U_m^\dagger = \mathcal{R}_{m+1} + \sigma'' \qquad \text{for all} \quad m \in \mathbb{N}. \tag{9}$$

Furthermore, setting for all $m \in \mathbb{N}$,

$$|n\rangle = |m+n, m\rangle, \qquad e_n = \lambda_{m+n} - \lambda_m, \qquad m \in \mathbb{N},$$

we obtain

$$a_m|n\rangle = \sqrt{e_n}|n-1\rangle, \qquad a_m^\dagger|n\rangle = \sqrt{e_{n+1}}|n+1\rangle, \qquad (H_m - \lambda_m)|n\rangle = e_n|n\rangle. \tag{10}$$

One can then deduce the CS for AHF as eigenstates of the annihilation operator a_m $(a_m|z\rangle = z|z\rangle)$ [36] in the following way:

$$|z\rangle = \mathcal{N}(|z|^2) \sum_{n\geqslant 0}^{\infty} \frac{z^n}{\sqrt{\varepsilon_n}}|n\rangle \qquad \mathcal{N}(|z|^2) = \left[\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\varepsilon_n}\right]^{-1/2} \tag{11}$$

for any z in the open disc $\mathcal{C}(O, \mathcal{R})$ with centre O and radius $\mathcal{R} = \limsup_{n\to\infty} \sqrt[n]{\varepsilon_n} \neq 0$ where

$$\varepsilon_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ e_1 e_2 \cdots e_n & \text{if} \quad n > 0. \end{cases}$$

3. Generalized coherent states for associated hypergeometric-type functions

This section aims at generalizing the CS (11) on the basis of similarity properties between the shape-invariance relation of a given system and the AHF. Indeed, as quoted in [18], the factorization operators A and $A^\dagger$, the ladder operators $B_\pm$, the shape-invariance parameter $\mathcal{R}(a_1)$ can be paired, respectively, to A_m, $A_m^\dagger$, a_m, $a_m^\dagger$, $\mathcal{R}_m$ or r_{m+n+1}, in [36].

Although the operators A_m and a_m do not possess inverses, one can define for them the so-called *right inverse* as [29]

$$A_m A_m^{-1} = 1, \qquad a_m a_m^{-1} = 1.$$

The right inverse a_m^{-1} of a_m can be paired with the right inverse B_-^{-1} of B_- [29–31] which is used for the generalization of the CS for shape-invariant systems [18]. In terms of a_m^{-1}, the CS (11) read

$$|z\rangle = \mathcal{N}(|z|^2) \sum_{n=0}^{\infty} \left(z a_m^{-1}\right)^n |0\rangle \tag{12}$$

since one can readily show that

$$\left(a_m^{-1}\right)^n |0\rangle = \frac{1}{\sqrt{\varepsilon_n}}|n\rangle. \tag{13}$$

From $[a_m, \mathcal{R}_m] = -\sigma'' a_m$, we can deduce

$$a_m \mathcal{R}_m = (\mathcal{R}_m - \sigma'')a_m, \qquad a_m f(\mathcal{R}_m) = f(\mathcal{R}_m - \sigma'')a_m, \tag{14}$$

for any analytic function f. The relation (9) can then be generalized as

$$U_m f(\mathcal{R}_m) U_m^\dagger = f(\mathcal{R}_{m+1} + \sigma''). \tag{15}$$

Using (8) and (15), we obtain

$$\left\{f(\mathcal{R}_m)a_m^{-1}\right\}^n = \left\{\prod_{k=0}^{n-1} f(\mathcal{R}_m + k\sigma'')\right\} a_m^{-n}. \tag{16}$$

Let us now define the generalized CS as

$$|z;\mathcal{R}_m\rangle = \sum_{n=0}^{\infty}\left\{zf(\mathcal{R}_m)a_m^{-1}\right\}^n|0\rangle. \tag{17}$$

The relation (17) can be rewritten as

$$|z;\mathcal{R}_m\rangle = \frac{1}{1 - zf(\mathcal{R}_m)a_m^{-1}}|0\rangle. \tag{18}$$

From (13) and (16), we deduce the general Glauber form [8] of (17)

$$|z;\mathcal{R}_m\rangle = \sum_{n=0}^{\infty}\frac{z^n}{h_n(\mathcal{R}_m)}|n\rangle, \tag{19}$$

where

$$h_n(\mathcal{R}_m) = \frac{\sqrt{\varepsilon_n}}{\prod_{k=0}^{n-1} f(\mathcal{R}_m + k\sigma'')}. \tag{20}$$

Using (14) one can prove that the CS (19) are eigenstates of a_m, that is

$$a_m|z;\mathcal{R}_m\rangle = zf(\mathcal{R}_m - \sigma'')|z;\mathcal{R}_m\rangle, \tag{21}$$

and verify the condition

$$\{a_m - zf(\mathcal{R}_m - \sigma'')\}\frac{\partial}{\partial z}|z;\mathcal{R}_m\rangle = f(\mathcal{R}_m - \sigma'')|z;\mathcal{R}_m\rangle. \tag{22}$$

Taking into account the fact that $\mathcal{R}_m$ is an operator which acts on the states $|n\rangle$ as

$$\mathcal{R}_m|n\rangle = [-(m+n)\sigma'' - \tau']|n\rangle = r_{m+n+1}|n\rangle,$$

we can rewrite the CS (19) as

$$|z;m\rangle = \sum_{n=0}^{\infty}\frac{z^n}{h_n(m)}|n\rangle, \tag{23}$$

where

$$h_n(m) = \frac{\sqrt{\varepsilon_n}}{\prod_{k=0}^{n-1} f(r_{m+n+1-k})} = \frac{\sqrt{\varepsilon_n}}{\prod_{k=0}^{n-1} f((k-m-n)\sigma'' - \tau')}. \tag{24}$$

The properties (21) and (22) become, respectively,

$$a_m|z;m\rangle = zf(r''_{m+n+2})|z;m\rangle, \tag{25}$$

$$\{a_m - zf(r_{m+n+2})\}\frac{\partial}{\partial z}|z;m\rangle = f(r_{m+n+2})|z;m\rangle. \tag{26}$$

Let us observe that the generalized coherent states (23) verify the label continuity requirement since the transformation $(z,m) \longrightarrow (z',m')$ is equivalent to the transformation of states $|z,m\rangle \longrightarrow |z',m'\rangle$. Let us examine the realization of the properties of overcompleteness, temporal stability and action identity for the CS (23) in the following subsections.

3.1. Normalization

It is easy to prove that the coherent state (23) can be normalized as

$$|z;m\rangle = \mathcal{N}(|z|^2;m)\sum_{n=0}^{\infty}\frac{z^n}{h_n(m)}|n\rangle \tag{27}$$

where the normalization factor reads

$$\mathcal{N}(x;m) = \left[\sum_{n=0}^{\infty}\frac{x^n}{|h_n(m)|^2}\right]^{-1/2}. \tag{28}$$

We can see that the inner product of two CS does not vanish

$$\langle z';m|z;m\rangle = \frac{\mathcal{N}(|z'|^2;m)\mathcal{N}(|z|^2;m)}{\mathcal{N}(z'^*z;m)^2}. \tag{29}$$

This shows that the CS are not mutually orthogonal. Considering the normalized form of the CS (27), the condition (25) becomes

$$a_m|z,m\rangle = zf(r_{m+n+2})\frac{\mathcal{N}(|z|^2;m+1)}{\mathcal{N}(|z|^2;m)}|z;m\rangle. \tag{30}$$

The radius of the convergence of the series which defines the normalization factor $\mathcal{N}(|z|^2;m)$ is given by $\mathcal{R} = \limsup_{n\to+\infty}\sqrt[n]{|h_n(m)|^2}$. The expressions of $h_n(m)$, $\mathcal{N}(|z|^2;m)$ and $\mathcal{R}$ depend on the choice of the analytic function f.

3.2. Overcompleteness

We assume the existence of a non-negative weight function ω so that the overcompleteness or resolution of identity holds

$$\int_{\mathbb{C}} \mathrm{d}^2z|z;m\rangle\langle z;m|\omega(|z|^2;m) = \mathbb{1}_{\mathcal{H}m} \tag{31}$$

where $\mathbb{1}_{\mathcal{H}m}$ is the identity operator in the Hilbert space $\mathcal{H}_m$ of the eigenstates of the operator H_m. Introducing (27) into (31), we can see, after straightforward computation of the angular integration, that the weight function ω must fulfil the condition

$$\int_0^{\infty}\mathrm{d}\rho\,\rho^n\mathcal{W}(\rho;m) = |h_n(m)|^2, \qquad \mathcal{W}(\rho;m) = \pi\mathcal{N}^2(\rho;m)\omega(\rho;m). \tag{32}$$

Here, we use the polar representation $z = r\,\mathrm{e}^{\mathrm{i}\phi}$; ρ stands for r^2. Therefore, the weight function ω is related to the undetermined moment distribution $\mathcal{W}(\rho,m)$, which is the solution of the Stieltjes moment problem with the moments given by $|h_n(m)|^2$. Following step by step [4], one can determine the measure $\omega(\rho,m)$. In the Fourier representation, $\mathcal{W}(\rho,m)$ is given by [18]

$$\mathcal{W}(\rho,m) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t\;\Phi(t;m)\,\mathrm{e}^{-\mathrm{i}\rho t}, \tag{33}$$

where

$$\phi(t;m) = \sum_{n=0}^{\infty}|h_n(m)|^2\frac{(\mathrm{i}t)^n}{n!}. \tag{34}$$

The explicit expression of $\mathcal{W}(\rho,m)$ depends on the choice of the analytic function f and can be worked out following standard handbooks of tabulated integrals [40–42].

3.3. Temporal stability

Let us consider the following transformation $f(\mathcal{R}_m) \longrightarrow \tilde{f}(\mathcal{R}_m) = f(\mathcal{R}_m)\,\mathrm{e}^{-\mathrm{i}\alpha(\mathcal{R}_m+\sigma'')}$, corresponding to $f(r_{m+n+1}) \longrightarrow f(r_{m+n+1})\,\mathrm{e}^{-\mathrm{i}\alpha r_{m+n}}$, α being a real constant. The parameter $h_n(m)$ in (24) becomes $h_n(m)\,\mathrm{e}^{\mathrm{i}\alpha e_n}$. Therefore, the coherent states $|z; m\rangle$ can be rewritten as

$$|z; \alpha, m\rangle = \mathcal{N}(|z|^2; m) \sum_{n=0}^{\infty} \frac{z^n}{h_n(m)}\,\mathrm{e}^{-\mathrm{i}\alpha e_n}|n\rangle \tag{35}$$

where e_n is the eigenstate of $H_m - \lambda_m$ given by $e_n = \sum_{k=1}^{n} r_{m+k}$.

Let us show that the CS $|z; \alpha, m\rangle$ fulfil the temporal stability condition. By temporal stability, we mean that the time evolution of any coherent state $|z; \alpha, m\rangle$ remains coherent. This condition is satisfied if [2]

$$\mathrm{e}^{-\mathrm{i}(H_m-\lambda_m)t}|z; \alpha, m\rangle = |z; \alpha + \omega t, m\rangle, \qquad \omega = \text{constant}.$$

We have

$$\mathrm{e}^{-\mathrm{i}(H_m-\lambda_m)t}|z; \alpha, m\rangle = \mathcal{N}(|z|^2; m) \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha e_n}\,\mathrm{e}^{-\mathrm{i}(H_m-\lambda_m)t}|n\rangle.$$

Since

$$(H_m - \lambda_m)|n\rangle = e_n|n\rangle,$$

we have

$$\mathrm{e}^{-\mathrm{i}(H_m-\lambda_m)t}|z; \alpha, m\rangle = \mathcal{N}(|z|^2; m) \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}(\alpha+t)e_n}|n\rangle.$$

Therefore,

$$\mathrm{e}^{-\mathrm{i}(H_m-\lambda_m)t}|z; \alpha, m\rangle = |z; \alpha + t, m\rangle \tag{36}$$

which ends the proof of the temporal stability of the CS $|z; \alpha, m\rangle$.

3.4. Action identity

We show now that the CS $|z; m\rangle$ satisfy the action identity. That means we can define the canonical action-angle variables J, ν such as [18]

$$\begin{cases} \langle H_m - \lambda_m\rangle = cJ, \qquad c = \text{constant}, \\ \dfrac{\partial\langle H_m - \lambda_m\rangle}{\partial J} = \dot{\nu}. \end{cases} \tag{37}$$

From the conjugate of (25) given by

$$\langle z; m|a_m^\dagger = \langle z; m|z^* f^*(r_{m+n+2}) \tag{38}$$

we have

$$\frac{\langle z; m|H_m - \lambda_m|z; m\rangle}{\langle z; m|z; m\rangle} = \frac{\langle z; m|a_m^\dagger a_m|z; m\rangle}{\langle z; m|z; m\rangle} = |zf(r_{m+n+2})|^2. \tag{39}$$

We can define a canonical action variable $J = \beta_m^* \beta_m$, where $\beta_m = \frac{1}{\xi} z f(r_{m+n+2})$, with ξ a non-zero constant, such as

$$\langle H_m - \lambda_m\rangle = cJ, \qquad c = |\xi|^2 \qquad \text{and} \qquad \frac{\partial\langle H_m - \lambda_m\rangle}{\partial J} = \dot{\nu}. \tag{40}$$

Then,

$$\dot{\nu} = c, \qquad \nu = ct + \alpha, \qquad \alpha = \text{constant}. \tag{41}$$

Thus, the action identity property is verified. Hence, the generalized coherent states we introduced for AHF fulfil all the properties needed to establish a close connection with classical and quantum states.

4. Examples

In this section, we apply the above formulae to compute the generalized coherent states for the associated Hermite, Laguerre, Jacobi and hypergeometric functions. The associated Legendre, Chebyshev and Gegenbauer functions can be deduced as special cases of associated Jacobi functions.

4.1. Coherent states for associated Hermite and Laguerre functions

For Hermite polynomials, the coefficients $\sigma(x)$ and $\tau(x)$ are

$$\sigma(x) = 1, \qquad \tau(x) = -x. \tag{42}$$

In the case of Laguerre polynomials, we have

$$\sigma(x) = x, \qquad \tau(x) = \alpha + 1 - x. \tag{43}$$

In both cases, $\sigma'' = 0$ and $\tau' = -1$. For any analytical function f, the operator $f(\mathcal{R}_m)$ acts on the states $|n\rangle$ as

$$f(\mathcal{R}_m)|n\rangle = f(1)|n\rangle. \tag{44}$$

For any integer l, $\lambda_l = -\frac{1}{2}l(l-1)\sigma'' - l\tau' = l$, so that $e_n = \lambda_{m+n} - \lambda_m = n$. Then,

$$\varepsilon_n = e_1 e_2 \cdots e_n = 1 \cdot 2 \cdot 3 \cdots n = n! \tag{45}$$

Taking into account the fact that $\sigma'' = 0$, we get

$$\prod_{k=0}^{n-1} f(\mathcal{R}_m + k\sigma'') = [f(\mathcal{R}_m)]^n. \tag{46}$$

From (20), (45) and (46), we obtain

$$h_n(\mathcal{R}_m) = \frac{\sqrt{n!}}{[f(\mathcal{R}_m)]^n} \qquad \text{and} \qquad h_n(m) = \frac{\sqrt{n!}}{[f(1)]^n}. \tag{47}$$

The normalization factor (28) becomes

$$\mathcal{N}(|z|^2; m) = \exp\left(-\tfrac{1}{2}|zf(1)|^2\right). \tag{48}$$

We are now able to derive from (27) the normalized CS for associated Hermite and Laguerre functions following (44):

$$|z; m\rangle = \exp\left(-\frac{1}{2}|zf(1)|^2\right) \sum_{n=0}^{\infty} \left(\frac{zf(1)}{\sqrt{n!}}\right)^n |n\rangle. \tag{49}$$

The inner product of two coherent states gives

$$\langle z'; m|z; m\rangle = \exp\left(-\tfrac{1}{2} f(1)^2 (|z'|^2 + |z|^2 - 2z'^{\star} z)\right). \tag{50}$$

In this case [18], the resolution of the identity is obtained with the measure $\omega(|z|^2, m) = [f(1)]^2/\pi$. Redefining the complex variable $z \longrightarrow zf(1)$, we can recover the usual expression of the bosonic CS [1]

$$|z; m\rangle = \exp\left[\frac{-|z|^2}{2}\right] \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{51}$$

$$\langle z'; m|z; m\rangle = \exp\left[-\frac{1}{2}(|z'|^2 + |z|^2 - 2z'^{\star} z)\right]. \tag{52}$$

4.2. Coherent states for associated Jacobi and hypergeometric functions

Here, $\sigma(x) = 1 - x^2$, $\tau(x) = (\beta - \alpha) - (\alpha + \beta + 2)x$ for Jacobi functions and $\sigma(x) = x(1 - x)$, $\tau(x) = (\alpha + 1) - (\alpha + \beta + 2)x$ for hypergeometric functions [39]. In both cases, $\sigma'' = -2$ and $\tau' = -(\alpha + \beta + 2)$. Denoting $\mu = (\alpha + \beta + 2)$ then $\lambda_l = l(l + \mu - 1)$, $e_n = n(2m + n + \mu - 1)$. Thus, we obtain

$$\varepsilon_n = n!\frac{\Gamma(2m + n + \mu)}{\Gamma(2m + \mu)}, \tag{53}$$

where Γ is the Euler gamma function. Let us set $f(\mathcal{R}_m) = c$, first, a constant function. Then,

$$\prod_{k=0}^{n-1} f(\mathcal{R}_m + k\sigma'') = c^n \tag{54}$$

and

$$h_n(\mathcal{R}_m) = \sqrt{\Gamma(n + 1)\frac{\Gamma(2m + n + \mu)}{c^{2n}\Gamma(2m + \mu)}}. \tag{55}$$

The normalization factor then reads [5]

$$\mathcal{N}(|z|^2; m) = [{}_0F_1(2m + \mu, |z|^2)]^{-1/2} = (\Gamma(2m + \mu)|z|^{1-2m-\mu} I_{2m+\mu-1}(2|z|))^{-1/2}, \tag{56}$$

where we take $c = 1$ without loss of generality. The coherent states read

$$|z; m\rangle = \frac{1}{\sqrt{{}_0F_1(2m + \mu, |z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(n + 1)(2m + \mu)_n}}|n\rangle, \qquad |z| < \infty \tag{57}$$

and the weight function is (see [42] p 196, formula (5.39))

$$\omega(|z|^2; m) = 2I_{2m+\mu-1}(2|z|)K_{2m+\mu-1}(2|z|) \tag{58}$$

where ${}_0F_1$ is the confluent hypergeometric function, I_ν and K_ν are the modified Bessel functions.

Now, let us choose the function f as

$$f(\mathcal{R}_m) = \sqrt{(-1/2\mathcal{R}_m)(-1/2\mathcal{R}_m)}. \tag{59}$$

Then,

$$\prod_{k=0}^{n-1} f(\mathcal{R}_m + k\sigma'') = \prod_{k=0}^{n-1} \sqrt{\left(k - \frac{\mathcal{R}_m}{2}\right)\left(k - \frac{\mathcal{R}_m}{2}\right)} \tag{60}$$

$$= \frac{\Gamma\left(n - \frac{\mathcal{R}_m}{2}\right)\Gamma\left(n - \frac{\mathcal{R}_m}{2}\right)}{\Gamma\left(-\frac{\mathcal{R}_m}{2}\right)\Gamma\left(-\frac{\mathcal{R}_m}{2}\right)}. \tag{61}$$

In this case, the coefficient $h_n(\mathcal{R}_m)$ reads

$$h_n(\mathcal{R}_m) = \left[\Gamma(n + 1)\frac{\Gamma(n + 2m + \mu)}{\Gamma(n + 2m + \mu)}\frac{\Gamma(-\mathcal{R}_m)}{\Gamma(n - \mathcal{R}_m)}\frac{\Gamma(-\mathcal{R}_m)}{\Gamma(n - \mathcal{R}_m)}\right]^{-1/2} \tag{62}$$

and

$$h_n({}_m) = \left[\Gamma(n + 1)\frac{\Gamma(n + 2m + \mu)}{\Gamma(n + 2m + \mu)}\frac{\Gamma(-(m + n) - \nu)}{\Gamma(n - (m + n) - \nu)}\frac{\Gamma(-(m + n) - \nu)}{\Gamma(n - (m + n) - \nu)}\right]^{-1/2}$$

$$= \sqrt{\Gamma(n + 1)\frac{(2m + \mu)_n}{[(-m - n - \nu)_n]^2}}, \tag{63}$$

where $\nu = \mu/2, \alpha_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ denotes the Pochammer function. The coefficient $h_n(m)$ is in the form $\sqrt{\frac{(b)_n}{(a_1)_n(a_2)_n}}$ such that $a_1 = a_2 < 0, a_1 + a_2 - b < 0$. Therefore [5], the general coherent states are defined on a unit circle $|z| = 1$. The normalization factor is a constant given in terms of the Gaussian hypergeometric function of unit argument as [5]

$$[\mathcal{N}(1,m)]^{-2} = \frac{\Gamma(2(m+\nu))}{(\Gamma(3m+n+3\nu))^2}. \tag{64}$$

The coherent states are then obtained as

$$|z;m\rangle = \left[\frac{\Gamma(2(m+\nu))}{(\Gamma(3m+n+3\nu))^2}\right]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(n+1)\frac{(2m+\mu)_n}{[(-m-n-\nu)_n]^2}}}|n\rangle. \tag{65}$$

For these last states, it appears difficult to prove the resolution of unity, since the identification of the measure leads to a cumbersome Stieltjes (Hausdorf) moment problem. This difficulty has been recently pointed out by Appl and Schiller [5].

5. Conclusion

In this paper, we have provided a generalization of coherent states for associated hypergeometric operators. Almost all of these states satisfy the standard properties of label continuity, overcompleteness, temporal stability and action identity, corresponding to the classical/quantum states. We have treated in detail the cases of associated Hermite, Laguerre and Jacobi hypergeometric functions. It is worth noting that the resolution of unity is quasi-unattainable for states involving no standard Stieltjes moment problem. This problem remains open and its resolution should help to enlarge the classes of coherent states for associated hypergeometric-type functions and should then improve our understanding of the quantum-to-classical transition state of physical systems.

In this study, the central point in the construction of coherent states remains the choice of the expression of the holomorphic function f in the definition of the generalized coherent states (17). The simpler and more appropriate the choice of f is, the easier is the deduction of the parameter function h_n (24), of the normalization constant $\mathcal{N}$ (27) and the resolution of the measure problem.

The hypergeometric operators are recovered from the associated hypergeometric one, setting $m = 0$, in their expressions. The same procedure could be applied to the expression of the corresponding coherent states constructed here using appropriate variable changes (see (49) and (51)).

Finally, this work presents a general formalism of constructing coherent states for associated hypergeometric-type functions, which was lacking in the literature. Here and in previous work related to the investigations of coherent states for second-order differential operator of mathematical functions, the possibility of constructing coherent states depends on the answers to the following crucial questions:

(i) Is the second-order differential operator factorizable or not in terms of lowering (annihilation) and raising (creation) first-order differential operators?

(ii) Do the eigenfunctions of the second-order differential operator satisfy a three-term recurrence relation?

(iii) Do the identified coherent states satisfy the standard properties of label continuity, overcompleteness, temporal stability and action identity?

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